Published for SISSA by Springer

Received: *November 4, 2014*
Accepted: *January 26, 2015*
Published: *February 13, 2015*

Soft theorem of $\mathcal{N} = 4$ SYM in Grassmannian formulation

Junjie Rao

Zhejiang Institute of Modern Physics, Zhejiang University, Hangzhou, 310027, P.R. China

E-mail: raojunjie@zju.edu.cn

Abstract: Inspired by the new soft theorem in gravity by Cachazo and Strominger, the soft theorem for color-ordered Yang-Mills amplitudes has also been identified by Casali. In this note, the same content of $\mathcal{N} = 4$ SYM using the Grassmannian formulation is studied. Explicitly, in the holomorphic soft limit, we reduce the n-particle amplitude in terms of Grassmannian contour integrations into the deformed $(n-1)$-particle amplitude by localizing k variables relevant to the n-th particle. Afterwards, the leading soft factor and sub-leading soft operator naturally emerge.

Keywords: Scattering Amplitudes, Supersymmetric gauge theory

ArXiv ePrint: 1410.5047

doi:10.1007/JHEP02(2015)087

JHEP02(2015)087

Contents

1 Introduction

Scattering amplitudes often have an universal soft behavior when the momentum of one external leg tends to zero. This soft limit can be traced back to the works [1–6]. Recently, a new soft theorem for tree level gravity amplitudes was studied in [7]. By using the BCFW construction and imposing the holomorphic soft limit, Cachazo and Strominger have proved that

$$\begin{aligned} &M_n\left(\lambda_n \to \varepsilon\lambda_n\right) \\ &= \frac{1}{\varepsilon^3}\sum_{a=1}^{n-2}\frac{\langle n-1,a\rangle^2[na]}{\langle n-1,n\rangle^2\langle na\rangle}M_{n-1}\left(\tilde\lambda_{n-1}\to\tilde\lambda_{n-1}+\varepsilon\frac{\langle an\rangle}{\langle a,n-1\rangle}\tilde\lambda_n,\,\tilde\lambda_1\to\tilde\lambda_1+\varepsilon\frac{\langle n-1,n\rangle}{\langle n-1,a\rangle}\tilde\lambda_n\right) \\ &\quad+O(\varepsilon^0), \end{aligned} \tag{1.1}$$

here for M_n and M_{n-1}, the unmentioned external kinematic data are un-deformed and we prefer to suppress them for conciseness.[1] Taylor expansion in ε exhibits three singular terms in orders ε^{-3}, ε^{-2} and ε^{-1}, while higher order terms in ε will be mixed with the less interesting $O(\varepsilon^0)$ parts.

A similar relation for tree level Yang-Mills amplitudes using the BCFW construction, proved by Casali [8], takes the form

$$\begin{aligned} &A_n\left(\lambda_n \to \varepsilon\lambda_n\right) \\ &= \frac{1}{\varepsilon^2}\frac{\langle n-1,1\rangle}{\langle n-1,n\rangle\langle n1\rangle}A_{n-1}\left(\tilde\lambda_{n-1}\to\tilde\lambda_{n-1}+\varepsilon\frac{\langle 1n\rangle}{\langle 1,n-1\rangle}\tilde\lambda_n,\,\tilde\lambda_1\to\tilde\lambda_1+\varepsilon\frac{\langle n-1,n\rangle}{\langle n-1,1\rangle}\tilde\lambda_n\right)+O(\varepsilon^0)\,, \end{aligned} \tag{1.2}$$

[1] Any amplitude in this note contains the delta function of momentum conservation.

where two singular terms in orders ε^{-2} and ε^{-1} appear after Taylor expansion. The mixing between higher order terms from the deformed A_{n-1} and $O(\varepsilon^0)$ parts also pertains to this case.

It is fruitful to study the same object from various viewpoints in physics because it will deepen our original understanding and reveal many hidden facts. Many related studies have been achieved including: the soft limit from Poincaré symmetry and gauge invariance [14, 15], Feynman diagrams approach [16], conformal symmetry approach in Yang-Mills theory [17], the soft limit in arbitrary dimensions [18–21], loop corrections of the soft limit [22–24, 26], string theory approach [25, 26], ambitwistor string approach [27, 28] and KLT approach [29].

In this note, a relatively novel way using the Grassmannian contour integral is shown to reproduce the soft theorem for amplitudes in $\mathcal{N}=4$ SYM. Relevant background on Grassmannian can be found in [9–12] and a brief review of key formulae related to N^{k-2}MHV amplitudes $A_n^{[k]}$ is given here. The first example is the NMHV (super)amplitude, written as

$$\begin{aligned} A_n^{[3]} &= \int_{\{f_6=\ldots=f_n=0\}} \frac{g_n^{[3]}}{(n-1)(1)(3)} \frac{1}{f_6\ldots f_n}, \\ g_n^{[3]} &= \prod_{j=6}^{n-1} (1\,2\,j)(2\,3\,j-1), \;\; f_l = (l-2\,l-1\,l)(l\,1\,2)(2\,3\,l-2), \end{aligned} \tag{1.3}$$

where (l) is short for the consecutive 3-minor $(l\,l+1\,l+2)$ in terms of c_{Ii}'s for the $k=3$ case. The integral symbol above denotes

$$\int d^{3(n-3)}c_{Ii}\,\delta^{2(n-3)}(\lambda_i-\lambda_I c_{Ii})\delta^{2\cdot 3}\left(\tilde\lambda_I+c_{Ii}\tilde\lambda_i\right)\delta^{4\cdot 3}(\tilde\eta_I+c_{Ii}\tilde\eta_i), \tag{1.4}$$

note that the supersymmetric content is not involved in solving c_{Ii}'s. There are $(n-5)$ actual integration variables to be localized[2] by $(n-5)$ f_i's in (1.3). Although the integrand in (1.3) is nothing but

$$\frac{1}{(1\,2\,3)(2\,3\,4)\ldots(n\,1\,2)}, \tag{1.5}$$

after the cancelation of non-consecutive minors between the numerator and denominator, for practical calculations (1.3) is adopted and the reason is: it is known that residue (or contour) integrations demand non-consecutive minors for physical outputs, while some consecutive minors are redundant. The advantage of the integrand in (1.3) is that all unwanted minors are 'lifted away' from the sequence of minors mapped to zero (i.e., they are only evaluated as residues at zero minors).

Another convenience of this integrand is that in process of the inverse soft operation (or 'add one particle at a time'), the soft factor is recovered in the soft limit. More explicitly, this factor in terms of minors is given via $I_n^{[3]}=I_{n-1}^{[3]}S_{(n-1)\to n}^{[3]}$ where $I_n^{[3]}$ is short for the integrand in (1.3), so that

$$S_{(n-1)\to n}^{[3]}=\frac{(n-2)'(n-1)'(n-2\,2\,3)}{(n-1)f_n}=\frac{(n-2)'(n-1)'}{(n-2)(n-1)(n)}, \tag{1.6}$$

[2] For general N^{k-2}MHV amplitudes, the $(k\times n)$ matrix c_{Ii} has GL(k) gauge invariance, hence there are $k(n-k)$ independent variables. Imposing $(2n-4)$ delta functions, there are $(k-2)(n-k-2)$ variables left to be fixed by contour integrations.

JHEP02(2015)087

here the prime means that the corresponding consecutive minor is for the $(n-1)$ case, namely $(n-2)' = (n-2\,n-1\,1)$ and $(n-1)' = (n-1\,1\,2)$. In the limit $\lambda_n \to \varepsilon\lambda_n$, three c_{In}'s are localized by two delta functions and one contour integration, which leads to

$$S^{[3]}_{(n-1)\to n} \to \frac{1}{\varepsilon^2}\frac{\langle n-1,1\rangle}{\langle n-1,n\rangle\langle n1\rangle}\,, \tag{1.7}$$

this is the desired soft factor. In [10] the leading singular term is mentioned, while in fact the sub-leading term is also automatically included, as will be demonstrated in this note.

Having explained the integrand for NMHV amplitudes, now let's present the universal structure of general N^{k-2}MHV amplitudes derived by the conjugation construction. Before this, one can compare the integrand for NMHV amplitudes (1.3), with the one for N^{2}MHV amplitudes given by

$$A^{[4]}_n = \int_{\{F_7=\ldots=F_n=0\}} \frac{g^{[4]}_n}{(n-1)(1)(3)}\frac{1}{F_7\ldots F_n}\,, \tag{1.8}$$

where

$$g^{[4]}_n = \prod_{j=7}^{n-1}(1\,2\,3\,j)(2\,3\,j-2\,j-1)(1\,j-2\,j-1\,j)\prod_{j=4}^{n-3}(1\,3\,j\,j+1)(1\,2\,j\,j+3)\,, \tag{1.9}$$

and $F_l = f_{l1}f_{l2}$ with

$$\begin{aligned} f_{l1} &= (l-3\,l-2\,l-1\,l)(l-3\,l\,1\,2)(l-3\,2\,3\,l-2)\,,\\ f_{l2} &= (1\,l-2\,l-1\,l)(1\,l\,2\,3)(1\,3\,l-3\,l-2)\,. \end{aligned} \tag{1.10}$$

The conjugation construction is implied in process of getting $g^{[4]}_n$ and F_l's from $g^{[3]}_n$ and f_l's, as well as transforming $(n-1)(1)(3)$ of $k=3$ into the same product but of $k=4$. By extending this construction, one can show that for general N^{k-2}MHV amplitudes,

$$A^{[k]}_n = \int_{\{F_{k+3}=\ldots=F_n=0\}} \frac{g^{[k]}_n}{(n-1)(1)(3)}\frac{1}{F_{k+3}\ldots F_n},\;\; F_i = f_{i1}\ldots f_{i,k-2}\,, \tag{1.11}$$

here each F_i is a product of $(k-2)$ f_{ij}'s which enforce $(k-2)$ minors to be zero, hence offsetting the $(k-2)$ variables brought by each newly added particle. Its details can be found in appendix A.

Returning to the inverse soft operation, we need to highlight a relation. Assume that the integrand $I^{[k]}_{n-1}$ is known, to get $I^{[k]}_n$ one simply needs to multiply $I^{[k]}_{n-1}$ by the inverse soft factor

$$S^{[k]}_{(n-1)\to n} = \frac{(n-k+1)'(n-k+2)'\ldots(n-1)'}{(n-k+1)(n-k+2)\ldots(n-1)(n)}\,, \tag{1.12}$$

which has been 'over-simplified' due to the cancelation of all non-consecutive minors between the numerator and denominator, same as what happens in (1.5), which is the simplicity in handling the soft limit: we need only focus on the consecutive minors.

After getting familiar with the Grassmannian formulation, we will show how to use it to reproduce the supersymmetric extension of (1.2) for all k's, especially two key components:

the overall soft factor, and the deformed anti-holomorphic spinor pair $(\tilde{\lambda}_{n-1}, \tilde{\lambda}_1)$ along with its deformed supersymmetric counterpart $(\tilde{\eta}_{n-1}, \tilde{\eta}_1)$, which nicely imitates the former.

This note is organized as follows. Section 2 reviews the inverse soft operation for NMHV amplitudes and proves the corresponding soft theorem. Section 3 explores the same aspects of N^2MHV and N^3MHV amplitudes and attempts to find the pattern for general k's. Section 4 provides a general proof of the soft theorem for N^{k-2}MHV amplitudes. Section 5 concludes with a few comments. Appendix A derives the general structure for N^{k-2}MHV amplitudes by using the conjugation construction. Appendix B explains why the unmentioned but possibly singular parts are actually regular in the soft limit.

2 NMHV amplitudes redux

In this section let's consider the simplest case, i.e., the NMHV amplitude formulated by

$$A_n^{[3]} = \int \frac{g_n^{[3]}}{(n-1)(1)(3)} \frac{1}{\underline{f_6} \cdots \underline{f_{n-1}} f_n}\,, \tag{2.1}$$

where to simplify the notation, underlines are used to indicate the zero factors for contour integrations. The global residue theorem manipulates above expression into

$$\begin{aligned} &\int \frac{g_n^{[3]}}{(n-1)(1)(3)} \frac{1}{\underline{f_6} \cdots \underline{f_{n-1}} f_n} \\ &\quad = -\int \frac{g_n^{[3]}}{\underline{(n-1)}(1)(3)} \frac{1}{\underline{f_6} \cdots \underline{f_{n-1}} f_n} - \int \frac{g_n^{[3]}}{(n-1)\underline{(1)}(3)} \frac{1}{\underline{f_6} \cdots \underline{f_{n-1}} f_n} \\ &\qquad - \int \frac{g_n^{[3]}}{(n-1)(1)\underline{(3)}} \frac{1}{\underline{f_6} \cdots \underline{f_{n-1}} f_n}\,, \end{aligned} \tag{2.2}$$

as will be explained in appendix B, among three terms above, only the $\underline{(n-1)}$ term has singular contribution in the soft limit $\lambda_n \to \varepsilon \lambda_n$.

To work out the integration, let's write the integral symbol explicitly as

$$\int d^{3(n-3)} c_{Ii}\, \delta^{2(n-3)}(\lambda_i - \lambda_I c_{Ii}) \delta^{2\cdot 3}\left(\tilde{\lambda}_I + c_{Ii} \tilde{\lambda}_i\right) \delta^{4\cdot 3}(\tilde{\eta}_I + c_{Ii} \tilde{\eta}_i). \tag{2.3}$$

In the Grassmannian formulation of c_{Ii}'s, one needs to choose a gauge. Among many choices, the following gauge provides maximal simplicity:

$$C = \begin{pmatrix} \ldots & 1 & 0 & c_{n-2,n} & 0 & c_{n-2,2} & \ldots \\ \ldots & 0 & 1 & c_{n-1,n} & 0 & c_{n-1,2} & \ldots \\ \ldots & 0 & 0 & c_{1n} & 1 & c_{12} & \ldots \end{pmatrix}, \tag{2.4}$$

where three columns $(n-2, n-1, 1)$ have been fixed to be a unit matrix.[3]

[3]The gauge choice is a bit different from the standard one where columns of negative helicities are often fixed, since under such a choice the supersymmetric counterpart can be integrated over most conveniently [9]. Here the three columns chosen to be fixed are not necessarily associated with negative helicities. But for the soft particle n, which has positive helicity, it is natural to have an unfixed column.

JHEP02(2015)087

Now we attempt to write the integrand possessing $\underline{(n-1)}$ in (2.2) into a form of the deformed $A^{[3]}_{n-1}$ multiplied by the soft factor, hence the integral can be split as

$$-\int d^{3(n-4)}c_{I\underline{i}}\,\delta^{2(n-4)}(\lambda_{\underline{i}}-\lambda_I c_{I\underline{i}})\delta^{2\cdot 3}\left(\tilde{\lambda}_I+c_{I\underline{i}}\tilde{\lambda}_{\underline{i}}+c_{In}\tilde{\lambda}_n\right)I^{[3]}_{n-1}$$
$$\times\left\{\int d^3c_{In}\delta^2(\lambda_n-\lambda_I c_{In})\frac{(n-2)'(n-1)'}{(n-2)\underline{(n-1)}(n)}\right\}, \qquad (2.5)$$

where $\underline{i}=1,\ldots,n-1$, and integrations over all c_{In}'s are collected inside the curly bracket. Note that the supersymmetric content has been set aside temporally since it does not provide localization constraints.

Keep in mind that the integrand inside the curly bracket is the inverse soft factor $S^3_{(n-1)\to n}$, which later turns into the soft factor when the residue is evaluated at $(n-1)=0$. To reveal this, let's compute the relevant minors as

$$(n-2)'=1, \qquad (n-1)'=c_{n-2,2}, \qquad (2.6)$$

$$(n-2)=c_{1n}, \quad (n-1)=-c_{n-2,n}, \quad (n)=-\begin{vmatrix} c_{n-2,n} & c_{n-2,2} \\ c_{n-1,n} & c_{n-1,2}\end{vmatrix}, \qquad (2.7)$$

and the integration measure is defined to be (be aware of the reversed cyclic order)[4]

$$\int d^3c_{In}\equiv\int dc_{1n}dc_{n-1,n}\int dc_{n-2,n}\,, \qquad (2.8)$$

since $(n-1)=0$ fixes $c_{n-2,n}=0$, hence the delta function $\delta^2(\lambda_n-\lambda_I c_{In})$ solves

$$c_{n-1,n}=\frac{\langle 1n\rangle}{\langle 1,n-1\rangle},\quad c_{1n}=\frac{\langle n-1,n\rangle}{\langle n-1,1\rangle}. \qquad (2.9)$$

Putting every piece together,

$$-\int d^3c_{In}\delta^2(\lambda_n-\lambda_I c_{In})\frac{(n-2)'(n-1)'}{(n-2)\underline{(n-1)}(n)}=\frac{1}{\langle n-1,1\rangle}\frac{1}{c_{n-1,n}c_{1n}}=\frac{\langle n-1,1\rangle}{\langle n-1,n\rangle\langle n1\rangle}, \qquad (2.10)$$

which matches the soft factor as promised.

Then what is going on in the rest parts of the Grassmannian integral? Recall (2.5), note that delta functions $\delta^{2(n-4)}\left(\lambda_{\underline{i}}-\lambda_I c_{I\underline{i}}\right)$ are unaffected, while delta functions $\delta^{2\cdot 3}\left(\tilde{\lambda}_I+c_{I\underline{i}}\tilde{\lambda}_{\underline{i}}+c_{In}\tilde{\lambda}_n\right)$ turns into

$$\delta^2\left(\tilde{\lambda}_{n-1}+c_{n-1,\underline{i}}\tilde{\lambda}_{\underline{i}}+c_{n-1,n}\tilde{\lambda}_n\right)\delta^2\left(\tilde{\lambda}_1+c_{1\underline{i}}\tilde{\lambda}_{\underline{i}}+c_{1n}\tilde{\lambda}_n\right)\delta^2\left(\tilde{\lambda}_2+c_{2\underline{i}}\tilde{\lambda}_{\underline{i}}+c_{2n}\tilde{\lambda}_n\right)$$
$$=\delta^2\left(\tilde{\lambda}_{n-1}+\frac{\langle 1n\rangle}{\langle 1,n-1\rangle}\tilde{\lambda}_n+c_{n-1,\underline{i}}\tilde{\lambda}_{\underline{i}}\right)\delta^2\left(\tilde{\lambda}_1+\frac{\langle n-1,n\rangle}{\langle n-1,1\rangle}\tilde{\lambda}_n+c_{1\underline{i}}\tilde{\lambda}_{\underline{i}}\right)\delta^2\left(\tilde{\lambda}_2+c_{2\underline{i}}\tilde{\lambda}_{\underline{i}}\right), \qquad (2.11)$$

where the third one is also unaffected. Now let's recover the supersymmetric content, and we find that it trivially imitates the expression above, namely

$$\delta^{4\cdot 3}(\tilde{\eta}_I+c_{Ii}\tilde{\eta}_i)=\delta^4\left(\tilde{\eta}_{n-1}+\frac{\langle 1n\rangle}{\langle 1,n-1\rangle}\tilde{\eta}_n+c_{n-1,\underline{i}}\tilde{\eta}_{\underline{i}}\right)\delta^4\left(\tilde{\eta}_1+\frac{\langle n-1,n\rangle}{\langle n-1,1\rangle}\tilde{\eta}_n+c_{1\underline{i}}\tilde{\eta}_{\underline{i}}\right)\delta^4(\tilde{\eta}_2+c_{2\underline{i}}\tilde{\eta}_{\underline{i}}). \qquad (2.12)$$

[4]The reason to choose this order will be explained in section 4.

Plug (2.11) back into (2.5) and also take (2.12) into account, we just recover the soft theorem for NMHV amplitudes in $\mathcal{N}=4$ SYM at tree level. Explicitly, (2.1) becomes

$$\begin{aligned} A_n^{[3]}(\lambda_n \to \varepsilon\lambda_n) = &\frac{1}{\varepsilon^2}\frac{\langle n-1,1\rangle}{\langle n-1,n\rangle\langle n1\rangle}A_{n-1}^{[3]}\Big((\tilde\lambda,\tilde\eta)_{n-1} \to (\tilde\lambda,\tilde\eta)_{n-1} \\ &+\varepsilon\frac{\langle 1n\rangle}{\langle 1,n-1\rangle}(\tilde\lambda,\tilde\eta)_n,\ (\tilde\lambda,\tilde\eta)_1 \to (\tilde\lambda,\tilde\eta)_1 + \varepsilon\frac{\langle n-1,n\rangle}{\langle n-1,1\rangle}(\tilde\lambda,\tilde\eta)_n\Big) \\ &+ (\text{pure regular parts}), \end{aligned} \tag{2.13}$$

where $(\tilde\lambda,\tilde\eta)_n$ is replaced by $\varepsilon(\tilde\lambda,\tilde\eta)_n$ to manifest the soft divergence. Here let's call the first term above 'the singular parts', but it in fact contains 'mixed regular parts' after Taylor expansion in ε. In contrast, the 'pure regular parts' do not involve the $1/\varepsilon^2$ prefactor. These pure regular parts correspond to terms whose A_L's are not 3-particle amplitudes in the BCFW construction, as the reader can look up in [7, 8]. To see the singular parts directly, one can expand the expression above as

$$A_n^{[3]}(\lambda_n \to \varepsilon\lambda_n) = \left(\frac{1}{\varepsilon^2}S^{(0)} + \frac{1}{\varepsilon}S^{(1)}\right)A_{n-1}^{[3]} + O(\varepsilon^0)\,, \tag{2.14}$$

where the leading soft factor and sub-leading soft operator are defined as

$$S^{(0)} \equiv \frac{\langle n-1,1\rangle}{\langle n-1,n\rangle\langle n1\rangle},\ S^{(1)} \equiv \frac{1}{\langle n-1,n\rangle}\left(\tilde\lambda_n^{\dot\alpha}\frac{\partial}{\partial\tilde\lambda_{n-1}^{\dot\alpha}} + \tilde\eta_n^A\frac{\partial}{\partial\tilde\eta_{n-1}^A}\right) + \frac{1}{\langle n1\rangle}\left(\tilde\lambda_n^{\dot\alpha}\frac{\partial}{\partial\tilde\lambda_1^{\dot\alpha}} + \tilde\eta_n^A\frac{\partial}{\partial\tilde\eta_1^A}\right), \tag{2.15}$$

note that at order ε^{-1} there are $\tilde\eta$ parts in addition to the $\tilde\lambda$ parts, which extends the sub-leading soft operator of (1.2) supersymmetrically.[5] Still, this supersymmetric content behaves exactly like its bosonic counterpart. Therefore we choose to suppress it in the following sections for brevity. One can easily recover this content by imitating the $\tilde\lambda$ part.

3 More extensions: N^{2}MHV and N^{3}MHV amplitudes

Having accomplished the simplest case, now we would like to explore the N^{2}MHV and N^{3}MHV amplitudes. From these two further examples, the pattern for general N^{k-2}MHV amplitudes, which will be revealed in section 4, starts to emerge.

To begin with the N^{2}MHV amplitude, recall (1.8) and (1.10), by applying the global residue theorem, the relevant integral is

$$\begin{aligned} \int \frac{g_n^{[4]}}{f'}\frac{1}{\underline{F_7}\ldots\underline{F_{n-1}}(\underline{f_{n1}}\,\underline{f_{n2}})} &= -\int \frac{g_n^{[4]}}{\underline{f'}}\frac{1}{\underline{F_7}\ldots\underline{F_{n-1}}(f_{n1}\underline{f_{n2}})} \\ &= -\int \frac{g_n^{[4]}}{\underline{(n-1)}(1)(3)}\frac{1}{\underline{F_7}\ldots\underline{F_{n-1}}f_{n1}\underline{(n-2)}(n)(\ldots)} - \ldots, \end{aligned} \tag{3.1}$$

where $f' = (n-1)(1)(3)$ for conciseness, we also remind the reader that $f_{n1} = (n-3)(\ldots)(\ldots)$ and $f_{n2} = (n-2)(n)(\ldots)$ where dots in parentheses denote the less important non-consecutive minors. In the second line above, among 3^2 choices of zero minors in f' and

[5] The same result given by the super BCFW construction can be found in [13].

f_{n2}, let's single out the term with $(n-1) = (n-2) = 0$, since other terms do not give singular contributions.

After the key integral is identified in (3.1), following the similar recipe and using (1.12), we split the integrand into the part of remaining $(n-1)$ particles and the part of inverse soft factor, as treated in (2.5). Now let's focus on

$$-\int d^4 c_{In} \delta^2(\lambda_n - \lambda_I c_{In}) \frac{(n-3)'(n-2)'(n-1)'}{-(n-3)\underline{(n-2)(n-1)}(n)}, \tag{3.2}$$

as mentioned before, the reason to assign $(n-2)$ to be the second zero minor is that this choice turns out to be the only singular contribution in the soft limit. Its proof for general k's is given in appendix B. Note that there is another minus sign appeared due to swapping the positions of $(n-1)$ and $(n-2)$, since by default $(n-1)$ locates at the first place in the sequence of zero minors. The latter fact is also true for all k's, as shown in appendix A.

To proceed, with the previous experience we choose the gauge

$$C = \begin{pmatrix} \cdots & c_{n-2,n-3} & 1 & 0 & c_{n-2,n} & 0 & 0 & c_{n-2,3} & \cdots \\ \cdots & c_{n-1,n-3} & 0 & 1 & c_{n-1,n} & 0 & 0 & c_{n-1,3} & \cdots \\ \cdots & c_{1,n-3} & 0 & 0 & c_{1n} & 1 & 0 & c_{13} & \cdots \\ \cdots & c_{2,n-3} & 0 & 0 & c_{2n} & 0 & 1 & c_{23} & \cdots \end{pmatrix}, \tag{3.3}$$

where four columns $(n-2, n-1, 1, 2)$ have been fixed to be a unit matrix. Hence the relevant minors are

$$(n-3)' = -c_{2,n-3}, \qquad (n-2)' = 1, \qquad (n-1)' = -c_{n-2,3}, \tag{3.4}$$

$$(n-3) = \begin{vmatrix} c_{1,n-3} & c_{1n} \\ c_{2,n-3} & c_{2n} \end{vmatrix}, \quad (n-2) = -c_{2n}, \quad (n-1) = -c_{n-2,n}, \; (n) = \begin{vmatrix} c_{n-2,n} & c_{n-2,3} \\ c_{n-1,n} & c_{n-1,3} \end{vmatrix}, \tag{3.5}$$

and the integration measure is

$$\int d^4 c_{In} = \int dc_{2n} dc_{1n} dc_{n-1,n} dc_{n-2,n} = \int dc_{1n} dc_{n-1,n} \int dc_{2n} \int dc_{n-2,n}. \tag{3.6}$$

Pay attention to the order of dc_{In}'s as they in fact anticommute. A little subtlety here is that we have to match the orders of dc_{In}'s and zero minors, namely $dc_{2n} dc_{n-2,n}$ must be associated with $(n-2)(n-1)$, otherwise a sign factor will arise due to altering the order of either dc_{In}'s or zero minors. But nicely, there is no such a worry in this case and that's the reason to adopt the reversed cyclic order for dc_{In}'s. Also note that we always leave the integrations over c_{1n} and $c_{n-1,n}$ to the last step, after performing the $(k-2)$ residue integrations.

The two integrations easily fix $c_{2n} = c_{n-2,n} = 0$, and using the remaining two delta functions in (3.2) we find the solution (2.9). Hence the final result is

$$\frac{1}{\langle n-1, 1\rangle} \frac{1}{c_{n-1,n} c_{1n}} = \frac{\langle n-1, 1\rangle}{\langle n-1, n\rangle \langle n1\rangle}, \tag{3.7}$$

and the anti-holomorphic spinor pair $\left(\tilde{\lambda}_{n-1}, \tilde{\lambda}_1\right)$ is deformed identically as (2.11), regardless of the increase of k. Not surprisingly, the soft theorem is again recovered for the N^2MHV case.

JHEP02(2015)087

Next we move on to the case of N^{3}MHV amplitudes, because it contains a non-trivial feature that cannot be seen in the former case. By applying the global residue theorem, the relevant integral is[6]

$$\int \frac{g_n^{[5]}}{f'} \frac{1}{\underline{F_8} \ldots \underline{F_{n-1}}(\underline{f_{n1}}\,\underline{f_{n2}f_{n3}})} = -\int \frac{g_n^{[5]}}{\underline{f'}} \frac{1}{\underline{F_8} \ldots \underline{F_{n-1}}(f_{n1}\underline{f_{n2}f_{n3}})}, \tag{3.8}$$

where $f' = (n-1)(1)(3)$. In the r.h.s. above, among 3^3 choices of zero minors in f', f_{n2} and f_{n3}, as you may guess, we single out the one picking $(n-1)$, $(n-2)$ and $(n-3)$ respectively, where following expressions of f_{nj}'s are used:

$$f_{n1} = (n-4)(\ldots)(\ldots), \;\; f_{n2} = (n-3)(\ldots)(\ldots), \;\; f_{n3} = (n-2)(n)(\ldots)\,. \tag{3.9}$$

Following the pattern of (3.2) and using (1.12), let's calculate

$$-\int d^5 c_{In} \delta^2(\lambda_n - \lambda_I c_{In}) \frac{(n-4)'(n-3)'(n-2)'(n-1)'}{(-)^2(n-4)\underline{(n-3)(n-2)(n-1)}(n)}, \tag{3.10}$$

the reason to assign $(n-3)$ to be the third zero minor is the same as previous. Here, a sign factor also arises when $(n-1)$ is pulled through $(n-3)(n-2)$, since only the zero minors care about their order while others trivially commute.

To proceed as trickily as before, we choose the gauge

$$C = \begin{pmatrix} \ldots & c_{n-2,n-4} & c_{n-2,n-3} & 1 & 0 & c_{n-2,n} & 0 & 0 & 0 & c_{n-2,4} & \ldots \\ \ldots & c_{n-1,n-4} & c_{n-1,n-3} & 0 & 1 & c_{n-1,n} & 0 & 0 & 0 & c_{n-1,4} & \ldots \\ \ldots & c_{1,n-4} & c_{1,n-3} & 0 & 0 & c_{1n} & 1 & 0 & 0 & c_{14} & \ldots \\ \ldots & c_{2,n-4} & c_{2,n-3} & 0 & 0 & c_{2n} & 0 & 1 & 0 & c_{24} & \ldots \\ \ldots & c_{3,n-4} & c_{3,n-3} & 0 & 0 & c_{3n} & 0 & 0 & 1 & c_{34} & \ldots \end{pmatrix}, \tag{3.11}$$

where five columns $(n-2, n-1, 1, 2, 3)$ have been fixed to be a unit matrix. Compare this choice with those in NMHV and N^{2}MHV cases, keen eyes will immediately see the pattern: we always fix k columns $(n-2, n-1, 1, 2, \ldots, k-2)$ to be a unit matrix. Hence the relevant minors are

$$(n-4)' = \begin{vmatrix} c_{2,n-4} & c_{2,n-3} \\ c_{3,n-4} & c_{3,n-3} \end{vmatrix}, \qquad (n-3)' = c_{3,n-3}, \;\; (n-2)' = 1, \;\; (n-1)' = c_{n-2,4}, \tag{3.12}$$

$$(n-4) = \begin{vmatrix} c_{1,n-4} & c_{1,n-3} & c_{1n} \\ c_{2,n-4} & c_{2,n-3} & c_{2n} \\ c_{3,n-4} & c_{3,n-3} & c_{3n} \end{vmatrix}, \;\; (n-3) = \begin{vmatrix} c_{2,n-3} & c_{2n} \\ c_{3,n-3} & c_{3n} \end{vmatrix},$$

$$(n-2) = c_{3n}, \qquad (n-1) = -c_{n-2,n}, \qquad (n) = -\begin{vmatrix} c_{n-2,n} & c_{n-2,4} \\ c_{n-1,n} & c_{n-1,4} \end{vmatrix}, \tag{3.13}$$

and the integration measure is

$$\int d^5 c_{In} = \int dc_{3n} dc_{2n} dc_{1n} dc_{n-1,n} dc_{n-2,n} = -\int dc_{1n} dc_{n-1,n} \int dc_{2n} dc_{3n} \int dc_{n-2,n}\,, \tag{3.14}$$

[6]This form for $k=5$ will be explained in appendix A.

where the order above is chosen to fit $(n-3)(n-2)(n-1)$. We must start the integrations from the rightmost, so the residue at $(n-3)=0$ must be evaluated after finishing $(n-2)$ and $(n-1)$. In this way the cancelation of all other un-localized $c_{I\underline{i}}$'s is guaranteed, as the relevant minors factorize with particular zero entries. This is a general pattern of N^{k-2}MHV amplitudes, but it only starts to emerge from the N^3MHV case.

As expected, the final result of (3.10) is

$$\frac{1}{\langle n-1,1\rangle}\frac{1}{c_{n-1,n}c_{1n}}=\frac{\langle n-1,1\rangle}{\langle n-1,n\rangle\langle n1\rangle}, \tag{3.15}$$

with $c_{2n}=c_{3n}=c_{n-2,n}=0$, spinor pair $(\tilde\lambda_{n-1},\tilde\lambda_1)$ is deformed identically as (2.11). Once more, the soft theorem is recovered for the N^3MHV case.

4 General N^{k-2}MHV amplitudes

In the previous case, we actually presume that N^{k-2}MHV amplitudes have a universal structure, where $(n-1)$ and (n) play special roles, namely

$$A_n^{[k]}=\int\frac{g_n^{[k]}}{f'}\frac{1}{F_{k+3}\dots F_n},\quad f'=(n-1)(1)(3),\ \ F_i=f_{i1}\dots f_{i,k-2}\,, \tag{4.1}$$

which is constructed by conjugation in appendix A. Applying the global residue theorem, yields

$$\int\frac{g_n^{[k]}}{f'}\frac{1}{\underline{F_{k+3}}\dots\underline{F_{n-1}}(\underline{f_{n1}f_{n2}\dots f_{n,k-2}})}=-\int\frac{g_n^{[k]}}{\underline{f'}}\frac{1}{\underline{F_{k+3}}\dots\underline{F_{n-1}}(f_{n1}\underline{f_{n2}}\dots\underline{f_{n,k-2}})}, \tag{4.2}$$

where each underlined F_i enforces all $(k-2)$ f_{ij}'s it contains to be zero, and each f_{ij} contributes one zero minor at a time respectively. The form of f_{nj}'s is given by

$$f_{n1}=(n-k+1)(\dots)(\dots),\ \ \dots\ \ f_{n,k-3}=(n-3)(\dots)(\dots),\ f_{n,k-2}=(n-2)(n)(\dots). \tag{4.3}$$

Among 3^{k-2} choices of zero minors in the r.h.s. of (4.2), we single out the one picking $(n-1)=(n-k+2)=(n-k+3)=\dots=(n-2)=0$, selected from f' and f_{nj}'s with $j=2,\dots,k-2$. This one is the only term that has singular contribution in the holomorphic soft limit.

To proceed, we choose the gauge

$$C=\begin{pmatrix}
\cdots & c_{n-2,n-k+1} & c_{n-2,n-k+2} & \cdots & c_{n-2,n-4} & c_{n-2,n-3} & 1 & 0 & c_{n-2,n} & 0 & 0 & \dots & 0 & 0 & c_{n-2,k-1} & \cdots\\
\cdots & c_{n-1,n-k+1} & c_{n-1,n-k+2} & \cdots & c_{n-1,n-4} & c_{n-1,n-3} & 0 & 1 & c_{n-1,n} & 0 & 0 & \dots & 0 & 0 & c_{n-1,k-1} & \cdots\\
\cdots & c_{1,n-k+1} & c_{1,n-k+2} & \cdots & c_{1,n-4} & c_{1,n-3} & 0 & 0 & c_{1n} & 1 & 0 & \dots & 0 & 0 & c_{1,k-1} & \cdots\\
\cdots & c_{2,n-k+1} & c_{2,n-k+2} & \cdots & c_{2,n-4} & c_{2,n-3} & 0 & 0 & c_{2n} & 0 & 1 & \dots & 0 & 0 & c_{2,k-1} & \cdots\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots\\
\cdots & c_{k-3,n-k+1} & c_{k-3,n-k+2} & \cdots & c_{k-3,n-4} & c_{k-3,n-3} & 0 & 0 & c_{k-3,n} & 0 & 0 & \dots & 1 & 0 & c_{k-3,k-1} & \cdots\\
\cdots & c_{k-2,n-k+1} & c_{k-2,n-k+2} & \cdots & c_{k-2,n-4} & c_{k-2,n-3} & 0 & 0 & c_{k-2,n} & 0 & 0 & \dots & 0 & 1 & c_{k-2,k-1} & \cdots
\end{pmatrix}, \tag{4.4}$$

JHEP02(2015)087

where k columns $(n-2, n-1, 1, 2, \ldots, k-3, k-2)$ have been fixed to be a unit matrix. Next let's calculate the following integral

$$-\int d^k c_{In} \delta^2(\lambda_n - \lambda_I c_{In}) \frac{(n-k+1)'(n-k+2)'(n-k+3)' \ldots (n-1)'}{(-)^{k-3}(n-k+1)\underline{(n-k+2)(n-k+3)} \ldots \underline{(n-2)(n-1)}(n)}, \tag{4.5}$$

which is obtained by extending (3.10) for a generic k after using (1.12). The sign factor in the denominator arises when $(n-1)$ is pulled through other $(k-3)$ zero minors. As previous, the only singular contribution is from the sequence of zero minors selected above.

In the chosen gauge, the relevant minors are

$$\begin{aligned}
(n-k+1)' &= (-)^{(k-3)\cdot 3} \begin{vmatrix} c_{2,n-k+1} & \cdots & c_{2,n-3} \\ \vdots & \ddots & \vdots \\ c_{k-2,n-k+1} & \cdots & c_{k-2,n-3} \end{vmatrix}, \\
(n-k+2)' &= (-)^{(k-4)\cdot 4} \begin{vmatrix} c_{3,n-k+2} & \cdots & c_{3,n-3} \\ \vdots & \ddots & \vdots \\ c_{k-2,n-k+2} & \cdots & c_{k-2,n-3} \end{vmatrix}, \\
&\vdots \\
(n-4)' &= (-)^{2(k-2)} \begin{vmatrix} c_{k-3,n-4} & c_{k-3,n-3} \\ c_{k-2,n-4} & c_{k-2,n-3} \end{vmatrix}, \\
(n-3)' &= (-)^{1(k-1)} c_{k-2,n-3}, \;\; (n-2)' = 1, \;\; (n-1)' = (-)^{k-1} c_{n-2,k-1},
\end{aligned} \tag{4.6}$$

as well as

$$\begin{aligned}
(n-k+1) &= (-)^{(k-2)\cdot 2} \begin{vmatrix} c_{1,n-k+1} & \cdots & c_{1,n-3} & c_{1n} \\ \vdots & \ddots & \vdots & \vdots \\ c_{k-2,n-k+1} & \cdots & c_{k-2,n-3} & c_{k-2,n} \end{vmatrix}, \\
(n-k+2) &= (-)^{(k-3)\cdot 3} \begin{vmatrix} c_{2,n-k+2} & \cdots & c_{2,n-3} & c_{2n} \\ \vdots & \ddots & \vdots & \vdots \\ c_{k-2,n-k+2} & \cdots & c_{k-2,n-3} & c_{k-2,n} \end{vmatrix}, \\
&\vdots \\
(n-4) &= (-)^{3(k-3)} \begin{vmatrix} c_{k-4,n-4} & c_{k-4,n-3} & c_{k-4,n} \\ c_{k-3,n-4} & c_{k-3,n-3} & c_{k-3,n} \\ c_{k-2,n-4} & c_{k-2,n-3} & c_{k-2,n} \end{vmatrix}, \\
(n-3) &= (-)^{2(k-2)} \begin{vmatrix} c_{k-3,n-3} & c_{k-3,n} \\ c_{k-2,n-3} & c_{k-2,n} \end{vmatrix}, \\
(n-2) &= (-)^{1(k-1)} c_{k-2,n}, \;\; (n-1) = -c_{n-2,n}, \\
(n) &= (-)^{k-2} \begin{vmatrix} c_{n-2,n} & c_{n-2,k-1} \\ c_{n-1,n} & c_{n-1,k-1} \end{vmatrix}.
\end{aligned} \tag{4.7}$$

JHEP02(2015)087

And the integration measure is

$$\begin{aligned}\int d^k c_{In} &= \int dc_{k-2,n}dc_{k-3,n}\ldots dc_{2n}dc_{1n}dc_{n-1,n}dc_{n-2,n}\\ &= (-)^{1+2+\ldots+(k-4)}\int dc_{1n}dc_{n-1,n}\int dc_{2n}\ldots dc_{k-3,n}dc_{k-2,n}\int dc_{n-2,n},\end{aligned}\tag{4.8}$$

note that we have reversed the order of $dc_{k-2,n}dc_{k-3,n}\ldots dc_{2n}$ to fit $(n-k+2)(n-k+3)\ldots(n-2)$, hence a sign factor arises. The $(k-2)$ residue integrations at $(n-k+2)=(n-k+3)=\ldots=(n-2)=(n-1)=0$ fix $(k-2)$ c_{In}'s to be zero when proceeded from right to left, while $c_{n-1,n}$ and c_{1n} are localized by the delta function $\delta^2(\lambda_n-\lambda_I c_{In})$ in the last step.

After that, (4.7) reduces to

$$\begin{aligned}
(n-k+1) &= (-)^{(k-2)\cdot 2}\begin{vmatrix} c_{1,n-k+1} & \cdots & c_{1,n-3} & c_{1n}\\ c_{2,n-k+1} & \cdots & c_{2,n-3} & \underline{c_{2n}}\\ \vdots & \ddots & \vdots & \vdots\\ c_{k-2,n-k+1} & \cdots & c_{k-2,n-3} & \underline{c_{k-2,n}}\end{vmatrix} = (-)^{(k-2)\cdot 2+(k-3)}c_{1n}\begin{vmatrix} c_{2,n-k+1} & \cdots & c_{2,n-3}\\ \vdots & \ddots & \vdots\\ c_{k-2,n-k+1} & \cdots & c_{k-2,n-3}\end{vmatrix},\\
(n-k+2) &= (-)^{(k-3)\cdot 3}\begin{vmatrix} c_{2,n-k+2} & \cdots & c_{2,n-3} & c_{2n}\\ c_{3,n-k+2} & \cdots & c_{3,n-3} & \underline{c_{3n}}\\ \vdots & \ddots & \vdots & \vdots\\ c_{k-2,n-k+2} & \cdots & c_{k-2,n-3} & \underline{c_{k-2,n}}\end{vmatrix} = (-)^{(k-3)\cdot 3+(k-4)}c_{2n}\begin{vmatrix} c_{3,n-k+2} & \cdots & c_{3,n-3}\\ \vdots & \ddots & \vdots\\ c_{k-2,n-k+2} & \cdots & c_{k-2,n-3}\end{vmatrix},\\
&\vdots\\
(n-4) &= (-)^{3(k-3)}\begin{vmatrix} c_{k-4,n-4} & c_{k-4,n-3} & c_{k-4,n}\\ c_{k-3,n-4} & c_{k-3,n-3} & \underline{c_{k-3,n}}\\ c_{k-2,n-4} & c_{k-2,n-3} & \underline{c_{k-2,n}}\end{vmatrix} = (-)^{3(k-3)+2}c_{k-4,n}\begin{vmatrix} c_{k-3,n-4} & c_{k-3,n-3}\\ c_{k-2,n-4} & c_{k-2,n-3}\end{vmatrix},\\
(n-3) &= (-)^{2(k-2)}\begin{vmatrix} c_{k-3,n-3} & c_{k-3,n}\\ c_{k-2,n-3} & \underline{c_{k-2,n}}\end{vmatrix} = (-)^{2(k-2)+1}c_{k-3,n}\,c_{k-2,n-3},\\
(n-2) &= (-)^{1(k-1)}c_{k-2,n},\quad (n-1) = -c_{n-2,n},\\
(n) &= (-)^{k-2}\begin{vmatrix} \underline{c_{n-2,n}} & c_{n-2,k-1}\\ c_{n-1,n} & c_{n-1,k-1}\end{vmatrix} = (-)^{k-1}c_{n-1,n}\,c_{n-2,k-1}.
\end{aligned}\tag{4.9}$$

Now the pattern is clear: $(n-1)=0$ fixes $c_{n-2,n}=0$, hence (n) factorizes into $(-)^{k-1}c_{n-1,n}\,c_{n-2,k-1}$. $(n-2)=0$ fixes $c_{k-2,n}=0$, hence $(n-3)$ factorizes into $(-)^{2(k-2)+1}c_{k-3,n}\,c_{k-2,n-3}$, and a further integration fixes $c_{k-3,n}=0$. Having $c_{k-2,n}=c_{k-3,n}=0$, $(n-4)$ again factorizes. This pattern will repeat to the 'top' minor $(n-k+1)$. Therefore the correct order to proceed residue integrations is crucial. Combine the result above with (4.6) and plug them back into (4.5), all un-localized $c_{I\underline{i}}$'s cancel, and we reach the longing answer:

$$\frac{\langle n-1,1\rangle}{\langle n-1,n\rangle\langle n1\rangle}\times \text{Sign}\,,\tag{4.10}$$

where

$$\text{Sign} = \frac{(-)(-)^{1+2+\ldots+(k-4)}(-)^{k-1}(-)^{k-1}\times(-)^{2(k-2)}\ldots(-)^{(k-3)\cdot 3}}{(-)^{k-3}(-)(-)^{k-1}(-)^{k-1}\times(-)^{2(k-2)+1}\ldots(-)^{(k-3)\cdot 3+(k-4)}(-)^{(k-2)\cdot 2+(k-3)}}=1,\tag{4.11}$$

which is not a coincidence, but a consequence of cautiously chosen conventions. Since $c_{2n} = c_{3n} = \ldots = c_{k-2,n} = c_{n-2,n} = 0$, spinor pair $(\tilde{\lambda}_{n-1}, \tilde{\lambda}_1)$ is deformed identically as (2.11). The soft theorem for general N^{k-2}MHV amplitudes is now proved.

5 Conclusion

In this note, we see that although it is already difficult to get explicit expressions for N^2MHV amplitudes by performing all residue integrations, let alone general N^{k-2}MHV amplitudes, still in the soft limit, with sufficient tricks one is able to find the soft theorem for all k's while keeping the Grassmannian contour integrations of $A^{[k]}_{n-1}$ unsolved. In proving this relation, a judicious choice of the Grassmannian gauge leads to considerable simplification. Besides, the irrelevant role of non-consecutive minors, which deserves to be emphasized, greatly reduces the complexity of the integrand structure.

Acknowledgments

The author is grateful to Prof. Bo Feng for his ardent guidance during the project. This work is supported by Chinese NSF funding under contracts No.11031005, No.11135006 and No.11125523.

A Conjugation construction of general N^{k-2}MHV amplitudes

Given an amplitude $A^{[k]}_n$, one can construct two other amplitudes: $A^{[k]}_{n+m}$ with m extra particles of positive helicity, and $A^{[k+l]}_n$ with l positive helicities flipped. It is easy to construct $A^{[k]}_{n+m}$ by applying the inverse soft operation ('add one particle at a time') successively. However, for $A^{[k+l]}_n$ it is not so straightforward and we need to use a trick of conjugation.

Assume that $k = 3$ and $n = 6 + l$, the conjugation gives

$$A^{[3+l]}_{6+l} = \overline{A^{[3]}_{6+l}}\,, \tag{A.1}$$

then the general $A^{[3+l]}_n$ can be obtained from $A^{[3+l]}_{6+l}$, which we named as the 'seed amplitude', since the inverse soft operation will grow it into amplitudes for all n's while fixing $k' = 3+l$.

Based on this observation, we now present an incomplete approach to construct amplitudes for all k's. It is incomplete because we will leave the unnecessary part, which involves overwhelming products of non-consecutive minors, unspecified in the derivation. The complete formula can be found in [12].

Let's start by rewriting (4.1) as

$$A^{[k]}_n = \int \frac{g^{[k]}_n}{[(n-1)(1)(3)]_k} \frac{1}{[F_{k+3}\ldots F_n]_k},\ [F_i]_k = [f_{i1}\ldots f_{i,k-2}]_k\,, \tag{A.2}$$

where $[\ldots]_k$ is a collective type label, for instance, $[abc]_k = a_k b_k c_k$, and it is introduced to distinguish 'the same functions' of different k's. Setting $k = 3$ and $n = 6 + l$, then the seed amplitude is

$$A^{[3+l]}_{6+l} = \int \frac{\overline{g^{[3]}_{6+l}}}{[(8+l)(4)(6)]_{3+l}} \frac{1}{\left[\,\overline{f_6}\ldots\overline{f_{6+l}}\,\right]_{3+l}}\,, \tag{A.3}$$

after recalling that the conjugate of $(l)_k = (l\,l+1\,\ldots\,l+k-1)$ is $(l+k)_{n-k} = (l+k\,l+k+1\,\ldots\,l-1)$. To return to the analogous form of (A.2), we use the cyclic invariance to perform the following shift of labels: $i \to i-3$, now

$$A_{6+l}^{[3+l]} = \int \frac{\hat{s}\,\overline{g_{6+l}^{[3]}}}{[(5+l)(1)(3)]_{3+l}} \frac{1}{\left[\hat{s}\,\overline{f_6}\ldots\hat{s}\,\overline{f_{6+l}}\,\right]_{3+l}}\,, \tag{A.4}$$

where the $\hat{s}$ operator denotes the cyclic shift. Define new variables via $\hat{s}\,\overline{g_{6+l}^{[3]}} \equiv g_{6+l}^{[3+l]}$, and

$$\hat{s}\,\overline{f_6}\ldots\hat{s}\,\overline{f_{6+l}} \equiv f_{6+l,1}\ldots f_{6+l,1+l} = F_{6+l}\,, \tag{A.5}$$

hence we achieve

$$A_{6+l}^{[3+l]} = \int \frac{g_{6+l}^{[3+l]}}{[(5+l)(1)(3)]_{3+l}} \frac{1}{[F_{6+l}]_{3+l}}, \tag{A.6}$$

extending from $(6+l)$ to generic n by using 'add one particle a at time' successively, yields

$$A_n^{[3+l]} = \int \frac{g_n^{[3+l]}}{[(n-1)(1)(3)]_{3+l}} \frac{1}{[F_{6+l}\ldots F_n]_{3+l}},\ \ [F_i]_{3+l} = [f_{i1}\ldots f_{i,1+l}]_{3+l}\,, \tag{A.7}$$

which returns to (A.2). From the initial amplitude $A_6^{[3]}$, we know that each f_{ij} contains three minors and this is true for all n's and k's, as implied by this construction. It is also confirmed by the first non-trivial example of N^{2}MHV amplitudes, as mentioned in section 1.

We have left the explicit expressions of f_{ij}'s unspecified, but for the sake of proving the soft theorem, one key fact must be clarified: the form of f_{nj}'s is given by

$$f_{n1} = [(n{-}k{+}1)(\ldots)(\ldots)]_k,\ \ \ldots\ \ f_{n,k-3} = [(n{-}3)(\ldots)(\ldots)]_k,\ \ f_{n,k-2} = [(n-2)(n)(\ldots)]_k. \tag{A.8}$$

Obviously the minor (n) plays a special role above, in addition to the special minor $(n-1)$. Also note that all consecutive minors involving column n, except $(n-1)$, must locate in $F_n = f_{n1}\ldots f_{n,k-2}$. For given generic n and k, the proof of this arrangement is the following.

By default, this arrangement trivially extends to the $(n+m)$ case while fixing k, then let's fix $n = 6+l$ and replace $k=3$ by $k' = 3+l$. Firstly (A.8) is valid for all n's in cases of $k=3$ and $k=4$, as confirmed in section 1. To extend it by induction, note that in process (A.5) of constructing F_{6+l} from $f_6\ldots f_{6+l}$, the operation of conjugation followed by label shift exactly preserves the minor labels, while $k=3$ is replaced by $k'=3+l$. Explicitly, we find the following form identical to (A.8), namely

$$f_{n,1} = [(n{-}k'{+}1)(\ldots)(\ldots)]_{k'},\ \ \ldots\ \ f_{n,k'-3} = [(n{-}3)(\ldots)(\ldots)]_{k'},\ \ f_{n,k'-2} = [(n-2)(n)(\ldots)]_{k'}, \tag{A.9}$$

where n and k' are kept instead of l. Therefore the proof is done.

The order of minors involving column n in (A.8) is $(n-k+1)(n-k+2)\ldots(n-2)(n)$, which justifies the order in (4.5).

B Pure regular parts

In this part, let's show that after using the global residue theorem, only one term has singular contribution. For the reader's convenience, we write (4.2) again here

$$\int \frac{g_n^{[k]}}{f'} \frac{1}{\underline{F_{k+3}} \cdots \underline{F_{n-1}} (\underline{f_{n1} f_{n2} \cdots f_{n,k-2}})} = - \int \frac{g_n^{[k]}}{\underline{f'}} \frac{1}{\underline{F_{k+3}} \cdots \underline{F_{n-1}} (f_{n1} \underline{f_{n2}} \cdots \underline{f_{n,k-2}})}, \tag{B.1}$$

with $f' = (n-1)(1)(3)$ and

$$f_{n1} = (n-k+1)(\ldots)(\ldots), \;\ldots\; f_{n,k-3} = (n-3)(\ldots)(\ldots), \; f_{n,k-2} = (n-2)(n)(\ldots). \tag{B.2}$$

It has been claimed that the only singular contribution in the soft limit is from the particular sequence of zero minors selected above, namely $(n-1)(n-k+2)(n-k+3)\ldots(n-2)$, while all other choices give regular terms.

To see why, let's recall the origin of soft divergence in the Grassmannian formulation. From section 4, we know that the $(k-2)$ residue integrations enforce $c_{2n} = c_{3n} = \ldots = c_{k-2,n} = c_{n-2,n} = 0$, leaving only $c_{n-1,n}$ and c_{1n} non-vanishing. Write (2.9) again here,

$$c_{n-1,n} = \frac{\langle 1n \rangle}{\langle 1, n-1 \rangle}, \; c_{1n} = \frac{\langle n-1, n \rangle}{\langle n-1, 1 \rangle}, \tag{B.3}$$

or equivalently,

$$\lambda_n = \lambda_{n-1} c_{n-1,n} + \lambda_1 c_{1n} \,. \tag{B.4}$$

It is natural to conceive that if there are some extra pieces besides λ_n on the l.h.s. of this equation, $c_{n-1,n}$ and c_{1n} would be 'protected' from vanishing in the limit $\lambda_n \to \varepsilon\lambda_n$, and hence the denominator involving $c_{n-1,n}$ and c_{1n} would not cause divergence since it is non-zero.

Favorably this is the right hint to catch. Rewrite the λ_n equation before localizing all c_{In}'s as

$$\lambda_n - \lambda_{\underline{I}} c_{\underline{I}n} = \lambda_{n-1} c_{n-1,n} + \lambda_1 c_{1n}, \tag{B.5}$$

where $\underline{I} = n-2, 2, 3, \ldots, k-2$. Any selected sequence of zero minors other than $(n-k+2)(n-k+3)\ldots(n-2)(n-1)$ fails to localize all $(k-2)$ $c_{\underline{I}n}$'s to be zero, since these are the only minors involving column n besides $(n-k+1)$ and (n), while the latter two are localized by the delta function $\delta^2(\lambda_n - \lambda_I c_{In})$ as always. Consider the extreme case where all $(k-2)$ $c_{\underline{I}n}$'s are not localized by these $(k-2)$ zero minors, then they must be localized by other constraints in the full Grassmannian integral. The relevant part in the integral involving c_{In}'s is

$$\begin{aligned} &\int d^k c_{In} \delta^2 (\lambda_n - \lambda_I c_{In}) \frac{\delta^{2\cdot k}\left(\tilde{\lambda}_I + c_{I\underline{i}}\tilde{\lambda}_{\underline{i}} + c_{In}\tilde{\lambda}_n\right)}{(n-k+1)(n-k+2)(n-k+3)\ldots(n-2)(n-1)(n)} \\ &= \frac{1}{\langle n-1, 1\rangle} \int d^{k-2} c_{\underline{I}n} \frac{\delta^{2\cdot(k-2)}\left(\tilde{\lambda}_{\underline{I}} + c_{\underline{I}\underline{i}}\tilde{\lambda}_{\underline{i}} + c_{\underline{I}n}\tilde{\lambda}_n\right) \delta^2\left(\tilde{\lambda}_{n-1} + c_{n-1,\underline{i}}\tilde{\lambda}_{\underline{i}} + c_{n-1,n}\tilde{\lambda}_n\right) \delta^2\left(\tilde{\lambda}_1 + c_{1\underline{i}}\tilde{\lambda}_{\underline{i}} + c_{1n}\tilde{\lambda}_n\right)}{(n-k+1)(n-k+2)(n-k+3)\ldots(n-2)(n-1)(n)}, \end{aligned} \tag{B.6}$$

where

$$c_{n-1,n} = \frac{\langle 1n \rangle - \langle 1\underline{I} \rangle c_{\underline{I}n}}{\langle 1, n-1 \rangle}, \; c_{1n} = \frac{\langle n-1, n \rangle - \langle n-1, \underline{I} \rangle c_{\underline{I}n}}{\langle n-1, 1 \rangle}. \tag{B.7}$$

Since $2(k-2) \geq k-2$, all $c_{\underline{I}n}$'s can be localized by delta functions selected from the $(k-2)$ constraints on their associated spinors. These values of $c_{\underline{I}n}$'s only depend on anti-holomorphic spinors, hence they are free from the holomorphic soft limit. When $c_{\underline{I}n}$'s are finite, $c_{n-1,n}$ and c_{1n} also remain finite even though $\lambda_n \to \varepsilon\lambda_n$, hence the possible divergence caused by these two variables safely dissolves in (B.6).

When $(k-2)$ $c_{\underline{I}n}$'s are partially localized, the argument above works analogously. This is why only one selected sequence of $(k-2)$ zero minors can cause soft divergence, as this choice delicately removes all 'protections' against pushing $c_{n-1,n}$ and c_{1n} to zero.

References

[1] F.E. Low, *Scattering of light of very low frequency by systems of spin 1/2*, Phys. Rev. **96** (1954) 1428 [INSPIRE].

[2] M. Gell-Mann and M.L. Goldberger, *Scattering of low-energy photons by particles of spin 1/2*, Phys. Rev. **96** (1954) 1433 [INSPIRE].

[3] S. Saito, *Low-energy theorem for Compton scattering*, Phys. Rev. **184** (1969) 1894 [INSPIRE].

[4] F.E. Low, *Bremsstrahlung of very low-energy quanta in elementary particle collisions*, Phys. Rev. **110** (1958) 974 [INSPIRE].

[5] S. Weinberg, *Photons and Gravitons in S-Matrix Theory: Derivation of Charge Conservation and Equality of Gravitational and Inertial Mass*, Phys. Rev. **135** (1964) B1049 [INSPIRE].

[6] S. Weinberg, *Infrared Photons and Gravitons*, Phys. Rev. **140** (1965) B516 [INSPIRE].

[7] F. Cachazo and A. Strominger, *Evidence for a New Soft Graviton Theorem*, arXiv:1404.4091 [INSPIRE].

[8] E. Casali, *Soft sub-leading divergences in Yang-Mills amplitudes*, JHEP **08** (2014) 077 [arXiv:1404.5551] [INSPIRE].

[9] N. Arkani-Hamed, F. Cachazo, C. Cheung and J. Kaplan, *A Duality For The S Matrix*, JHEP **03** (2010) 020 [arXiv:0907.5418] [INSPIRE].

[10] N. Arkani-Hamed, J. Bourjaily, F. Cachazo and J. Trnka, *Unification of Residues and Grassmannian Dualities*, JHEP **01** (2011) 049 [arXiv:0912.4912] [INSPIRE].

[11] D. Nandan, A. Volovich and C. Wen, *A Grassmannian Etude in NMHV Minors*, JHEP **07** (2010) 061 [arXiv:0912.3705] [INSPIRE].

[12] J.L. Bourjaily, J. Trnka, A. Volovich and C. Wen, *The Grassmannian and the Twistor String: Connecting All Trees in $N=4$ SYM*, JHEP **01** (2011) 038 [arXiv:1006.1899] [INSPIRE].

[13] Z.-W. Liu, *Soft theorems in maximally supersymmetric theories*, arXiv:1410.1616 [INSPIRE].

[14] J. Broedel, M. de Leeuw, J. Plefka and M. Rosso, *Constraining subleading soft gluon and graviton theorems*, Phys. Rev. **D 90** (2014) 065024 [arXiv:1406.6574] [INSPIRE].

[15] Z. Bern, S. Davies, P. Di Vecchia and J. Nohle, *Low-Energy Behavior of Gluons and Gravitons from Gauge Invariance*, *Phys. Rev.* **D 90** (2014) 084035 [arXiv:1406.6987] [INSPIRE].

[16] C.D. White, *Diagrammatic insights into next-to-soft corrections*, *Phys. Lett.* **B 737** (2014) 216 [arXiv:1406.7184] [INSPIRE].

[17] A.J. Larkoski, *Conformal Invariance of the Subleading Soft Theorem in Gauge Theory*, *Phys. Rev.* **D 90** (2014) 087701 [arXiv:1405.2346] [INSPIRE].

[18] B.U.W. Schwab and A. Volovich, *Subleading Soft Theorem in Arbitrary Dimensions from Scattering Equations*, *Phys. Rev. Lett.* **113** (2014) 101601 [arXiv:1404.7749] [INSPIRE].

[19] N. Afkhami-Jeddi, *Soft Graviton Theorem in Arbitrary Dimensions*, arXiv:1405.3533 [INSPIRE].

[20] M. Zlotnikov, *Sub-sub-leading soft-graviton theorem in arbitrary dimension*, *JHEP* **10** (2014) 148 [arXiv:1407.5936] [INSPIRE].

[21] C. Kalousios and F. Rojas, *Next to subleading soft-graviton theorem in arbitrary dimensions*, *JHEP* **01** (2015) 107 [arXiv:1407.5982] [INSPIRE].

[22] Z. Bern, S. Davies and J. Nohle, *On Loop Corrections to Subleading Soft Behavior of Gluons and Gravitons*, *Phys. Rev.* **D 90** (2014) 085015 [arXiv:1405.1015] [INSPIRE].

[23] S. He, Y.-t. Huang and C. Wen, *Loop Corrections to Soft Theorems in Gauge Theories and Gravity*, *JHEP* **12** (2014) 115 [arXiv:1405.1410] [INSPIRE].

[24] F. Cachazo and E.Y. Yuan, *Are Soft Theorems Renormalized?*, arXiv:1405.3413 [INSPIRE].

[25] B.U.W. Schwab, *Subleading Soft Factor for String Disk Amplitudes*, *JHEP* **08** (2014) 062 [arXiv:1406.4172] [INSPIRE].

[26] M. Bianchi, S. He, Y.-t. Huang and C. Wen, *More on Soft Theorems: Trees, Loops and Strings*, arXiv:1406.5155 [INSPIRE].

[27] T. Adamo, E. Casali and D. Skinner, *Perturbative gravity at null infinity*, *Class. Quant. Grav.* **31** (2014) 225008 [arXiv:1405.5122] [INSPIRE].

[28] Y. Geyer, A.E. Lipstein and L. Mason, *Ambitwistor strings at null infinity and (subleading) soft limits*, *Class. Quant. Grav.* **32** (2015) 055003 [arXiv:1406.1462] [INSPIRE].

[29] Y.-J. Du, B. Feng, C.-H. Fu and Y. Wang, *Note on Soft Graviton theorem by KLT Relation*, *JHEP* **11** (2014) 090 [arXiv:1408.4179] [INSPIRE].

JHEP02(2015)087